**REPORT**

by an external referee on the doctoral thesis by

**Jan ŠAFAŘÍK**

"WEAKLY DELAYED SYSTEMS OF LINEAR DISCRETE EQUATIONS IN R3"

submitted to obtain a Ph.D. degree,

at Brno University of Technology, Faculty of Electrical Engineering and Communication,

Department of Mathematics, 2018, 104 pp.

The thesis by Jan Šafařík, "Weakly Delayed Systems of Linear Discrete Equations in R3", is devoted to important problems of applied mathematics such as investigation of dynamics of systems of difference equations of a delayed type with after-effect, and in particular, to problems concerning the representation of solutions of linear autonomous systems with what is called a "weak delay".

Dynamical systems, expressed by functional differential equations, have recently found wide dissemination. If processes of mechanics and classical physics are described by Newton laws and occur "instantaneously", the analysis of processes in ecology, economy or dynamics of populations shows that the results of previous decisions come with a certain delay. In the population dynamics, the delay represents the time to maturity; in economy, the time needed to make and implement a decision. Thus, investigating a **topical** subject, the doctoral thesis is of great importance for applications.

Already in the monograph by J. Hale "Theory of Functional Differential Equations", an interesting phenomena was described. The spectrum of linear differential equations forms a countable set and, consequently, the equations have a countable set of linearly independent solutions. However, there are equations (and systems of equations), such that their spectrum is a finite set and their solutions, after a finite number of steps, "stick together". The resulting set of solutions is equivalent to the set of solutions of equations without delay. Such differential equations were then called "differential equations with weak delay". It is clear that similar effects can occur in the theory of difference equations with delay as well. Nowadays, the theory of difference equations with delay is being intensively developed. Such equations can be transformed into systems of difference equations without delay. However, such transformation matrices describing the processes corresponding to phase coordinates with different delays of argument "amalgamate into a single matrix" so that it is no longer possible to analyse the impact of each separate delay.

The doctoral thesis consists of four chapters (the first chapter being an introduction, the fourth one bringing conclusions), references (Bibliography) and list of used symbols (List of symbols and abbreviations).

The **first chapter** is of a preliminary nature having 15 pages (pp. 8 – 22). It contains the basics of linear algebra, particularly those related to the theory of matrices and eigenvalues, Laplace and Hamilton-Cayley theorems. The definitions of "difference systems with a weak delay" are given emphasizing the principal difference between differential and difference systems with weak delay. Differential systems with delay are systems in an infinite-dimensional Banach space. In the differential systems with weak delay, there is a dependence on the delay. The dimensions of systems are preserved.

Difference systems with delay are finite-dimensional systems as well. However, the principle of continuous dependence on the delay does not hold in them. Although they can be transformed into systems without delay, their dimension grows immensely.

The first part brings the definition of "difference systems with weak delay", formulating weak-delay conditions for the two- and three-dimensional cases (Theorems 1.4.1 and Theorem 1.4.2). It contains some assertions on the relation between weak delay and the nilpotency of matrices (Theorem 1.4.4). The conditions for weak delay in three-dimensional systems are formulated in Theorem 1.4.6.

Chapters Two and Three are central to the thesis.

**By its length and content, Chapter Two (pp. 23-46) has a preliminary character.** It considers matrices in a three-dimensional space with different eigenvalue types (different real, complex, multiple real). The Jordan forms are listed for all the above cases. The conditions are derived for each of the above cases. Part 2.1.1. derives the conditions for a weak delay if the eigenvalues are real and distinct, parts 2.1.2, 2.1.3. derive conditions for weak delay for double eigenvalues, parts 2.1.4, 2.1.5, 2.1.6 do it for triple eigenvalues, and part 2.1.7 for one real and two complex conjugate eigenvalues.

Despite the uniformity of the results derived, it should be noted that they required very complex and tedious transformations to be carried out. These demonstrate the doctoral student's thoroughness and mathematical skillfulness. The examples given demonstrate that the theoretical results derived are valid.

**Being a continuation of Chapter Two, Chapter Three is actually central to the thesis.** While in Chapter Two, conditions for weak delay were formulated, Chapter Three brings the forms of difference systems without delay as derived from systems with weak delay in three-dimensional space thanks to the increase in dimension. Theorem 3.1.1. presents theoretical results on the correspondence between systems with delay and non-delayed systems without delay but with high-dimension.

Next, a detailed analysis is performed of all of possible seven cases.

In my opinion, **the most interesting and essential** scientific result derived in the thesis is the method of transformation of systems with weak delay into corresponding systems without delay, derived in Chapter Three. Although the patterns of the proofs of all the seven cases considered are identical, the derivation of the results is technically complicated. The author carried out exceptionally complicated transformations receiving good results.

All the results obtained by the author and presented for the defence of the doctoral thesis are new. The credibility of the results obtained is based on rigorous mathematical proofs and on the fact that the results do not contradict the previous results.

The results obtained by Jan Šafařík in the thesis are important both for the theory and for applications. They further develop the theory of difference dynamical systems and difference systems with weak delay in particular.

The main results have been published in papers and presented at conferences. The doctoral thesis has been well commended. The short version of the Ph.D. thesis fully and adequately reflects the content of the thesis.

The good form of the doctoral thesis should also be mentioned. The results are presented in a concise and precise manner. The proofs of the results are exact with convincing arguments. First, a survey is given of the previous results with references. The list of references is complete and well reflects the current state of the problem.

After analyzing the entire doctoral thesis, I conclude that the scientific qualification of Jan Šafařík meets the requirements of the Ph.D. degree.

Comments to the thesis are the following. Formulas (1.1), (1.2), (1.3) denote systems of an identical form. This can cause difficulties in reading the thesis.

Example 3.2.2 (pp. 50-51), on the correspondence between the eigenvalues of systems with delay and of those without delay, is not fully comprehensible.

Based on the above analysis, I conclude that:

- the author's work on the doctoral thesis, ``Weakly Delayed Systems of Linear Discrete Equations in R3'', has been self-contained;
- the author has achieved important results in the theory of difference equations with delay.

It is my opinion that the doctoral thesis, "Weakly Delayed Systems of Linear Discrete Equations in R3'', based on the quantity of the investigation carried out, the topicality of its subject, the scientific level, quantity, and quality of the papers published by its author, meets all the necessary requirements and its author, Jan Šafařík, deserves to be awarded the degree of Ph.D.

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